ISRAEL JOURNAL OF MATHEMATICS 87 (1994), 1–8

THE UNIFORM NONEQUIVALENCE OF L_p AND l_p

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ABSTRACT

We prove that the Banach spaces $\mathbf{L}_p$ and $\mathbf{l}_p$ are uniformly nonequivalent for any $p > 2$. This result complements the well-known similar theorem of Bourgain for the case $p < 2$.

1. Introduction

The problem of topological classification of Banach spaces has a long history. In 1955 M. Kadec [4] proved that any two separable Banach spaces are homeomorphic to each other. So, the natural problem of the uniform classification of separable Banach spaces arises.

The Banach spaces $\mathbf{L}_p$ and $\mathbf{l}_p$ play a great role in functional analysis for $1 \leq p \leq \infty$ (see Section 2). It is clear that for $p = 2$, $\mathbf{L}_2$ and $\mathbf{l}_2$, being Hilbert spaces, are linearly isomorphic. In 1964 J. Lindenstrauss [5] proved that if $p \neq q$ and $\max\{p, q\} \geq 2$, then $\mathbf{l}_p$ and $\mathbf{l}_q$ are not uniformly homeomorphic. In 1969 P. Enflo [3] completed this result for any $p \neq q$. His proof works both for $\mathbf{L}_p$, $\mathbf{L}_q$ and for $\mathbf{l}_p$, $\mathbf{l}_q$. In 1986 Bourgain [2] proved that for $p < 2$, $\mathbf{L}_p$ and $\mathbf{l}_p$ are not uniformly homeomorphic.

In the present paper we complete these results by proving that $\mathbf{l}_p$ and $\mathbf{L}_p$ are uniformly nonequivalent when $p > 2$. I thank J. Lindenstrauss, who attracted my attention to this problem, and Y. Benyamini and J. Donin for their invaluable help.

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* Prof. Gorelik was killed in a car accident on September 23, 1993.
Received September 9, 1993

2. Main result

We will consider the linear space $\mathbf{l}_p$ of all sequences $x = \{x_i\}$ for which $\sum |x_i|^p < \infty$ with the norm $||x|| = (\sum |x_i|^p)^{1/p}$ and the space $\mathbf{L}_p$ of all classes of measurable functions g on the closed interval $[0,1]$ for which $\int |g|^p < \infty$ with the norm $||g|| = \left(\int |g|^p\right)^{1/p}$. The main result of the paper is the following

THEOREM 1: *For any $p > 2$ the spaces $\mathbf{L}_p$ and $\mathbf{l}_p$ are not uniformly homeomorphic.*

From now on we will use the following notations. Let $f\colon U \to V$ be a mapping of Banach spaces. If $x \in U$ then we denote $x' = f(x)$. For $x, y \in U$ we denote by xy the distance between x and y; similarly $x'y'$ is a distance between points in V.

It is known (see e.g. [1]) that every uniform homeomorphism between two Banach spaces $U \to V$ satisfies the following condition. For any $c > 0$ there exists $L = L(c)$ such that for any $x, y \in U$

$$xy > c \Longrightarrow \frac{1}{L} < \frac{x'y'}{xy} < L, \tag{2.1}$$

We will call this condition the double-sided Lipshitz condition for large distances (DLL). Hence, Theorem 1 follows from

THEOREM 2: *For any $p > 2$ there is no DLL-homeomorphism $f\colon \mathbf{l}_p \to \mathbf{L}_p$.*

We will prove this theorem by contradiction. So, we suppose that there exists a DLL-homeomorphism $f\colon \mathbf{l}_p \to \mathbf{L}_p$, and will get a contradiction.

3. Definition of parameters

Given a DLL-homeomorphism $f\colon \mathbf{l}_p \to \mathbf{L}_p$, we define the following parameters:

(1) L: $L = L(1)$ is the Lipshitz constant for the distances, which are greater than 1 (see (2.1)).
(2) δ: $\delta > 0$ is such that $\delta^p < 1/2$, and $\delta^{p/2-1} < 1/6pL^2$.
(3) q: $q = \delta^p/(1+\delta^p)$.
(4) n: a natural number satisfying the following inequality, $(1+q)^n > L^2$.
(5) $R = 2/\delta$.

4. Some geometric properties of $\mathbf{l}_p$ and $\mathbf{L}_p$

Our proof of Theorem 2 has the following scheme:

Lemma 2 $\Longrightarrow$ Lemma 1 $\Longrightarrow$ Theorem 2.

In this section we formulate Lemma 1, prove the implication **Lemma 1 $\Rightarrow$ Theorem 2**, after it we formulate Lemma 2, and prove the second implication **Lemma 2 $\Rightarrow$ Lemma 1**. Lemma 2 will be proved in the next sections.

LEMMA 1: *For any $A, B \in \mathbf{l}_p$, such that $AB > R$, there exists $C \in \mathbf{l}_p$, such that*

$$\frac{\max\{A'C', B'C'\}}{A'B'} : \frac{\max\{AC, BC\}}{AB} > 1 + q,$$

and $AC, BC > 1/4 \cdot AB$.

In particular we obtain

COROLLARY: *For any $A, B \in \mathbf{l}_p$, such that $AB > R$, there exist $C, D \in \mathbf{l}_p$, such that*

$$\frac{C'D'}{A'B'} > \frac{CD}{AB} \cdot (1 + q),$$

and $CD > 1/4 \cdot AB$.

Proof: Put $D = A$ if $A'C' = \max\{A'C', B'C'\}$, and put $D = B$ otherwise. The Corollary is proved. ∎

Proof of the implication **Corollary $\Rightarrow$ Theorem 2**: Take $A_1, B_1 \in \mathbf{l}_p$, such that $A_1B_1 > 4^n \cdot R$. By the corollary there exist $A_2, B_2 \in \mathbf{l}_p$, such that $A_2B_2 > 1/4 \cdot A_1B_1 > 4^{n-1}R$, and

$$\frac{A_2'B_2'}{A_1'B_1'} > \frac{A_2B_2}{A_1B_1} \cdot (1 + q).$$

Repeating this procedure n times, we find $A_{n+1}, B_{n+1} \in \mathbf{l}_p$ so that $A_{n+1}B_{n+1} > R$, and

$$\frac{A_{n+1}'B_{n+1}'}{A_1'B_1'} : \frac{A_{n+1}B_{n+1}}{A_1B_1} > (1 + q)^n > L^2.$$

On the other hand, since $A_1B_1, A_{n+1}B_{n+1} > R > 1$, it follows that $A_{n+1}'B_{n+1}' < L \cdot A_{n+1}B_{n+1}$ and $A_1'B_1' > A_1B_1/L$; so

$$\frac{A_{n+1}'B_{n+1}'}{A_1'B_1'} : \frac{A_{n+1}B_{n+1}}{A_1B_1} < L^2.$$

This contradiction proves the theorem. ∎

The proof of Lemma 1 is based on Lemma 2. We will use the following

Definition: Let X be a Banach space, $A, B \in X$, and let $\alpha > 0$ be a constant. Then $C \in X$ is called an **α-midpoint of A and B** if

$$\max\{AC, BC\} < AB/2 \cdot (1+\alpha).$$

Now we can formulate

LEMMA 2: *Let $A, B \in \mathbf{l}_p$ be such that $AB > R$; $A^*, B^* \in \mathbf{L}_p$ such that $AB/L < A^*B^* < L \cdot AB$. Denote $M \subset \mathbf{l}_p$ the set of all δ^p-midpoints of A, B, and $N \subset \mathbf{L}_p$ the set of all $2\delta^p$-midpoints of A^*, B^*. Then $f(M) \not\subset N$:*

Proof of the implication **Lemma 2 $\Rightarrow$ Lemma 1***:* Let $A, B \in \mathbf{l}_p$ be such that $AB > R$. It follows from Lemma 2 that for $A^* = A'$ and $B^* = B'$ there exists a point $C \in \mathbf{l}_p$ such that $C \in M$, and $C' \notin N$. Therefore

$$\frac{\max\{A'C', B'C'\}}{A'B'} : \frac{\max\{AC, BC\}}{AB} > \frac{1+2\delta^p}{1+\delta^p} = 1 + q\ .$$

On the other hand,

$$\min\{AC, BC\} \geq AB - \max\{AC, BC\} \geq \frac{AB}{2} \cdot (1-\delta^p) \geq \frac{AB}{4}.$$

Hence, Lemma 1 follows from Lemma 2. ∎

Before passing to the proof of Lemma 2 in the next sections, we make some reductions.

By standard approximation we can assume that A and B have only finitely many nonzero coordinates, and that the function $B^* - A^*$ never vanishes. By composing f with translations in $\mathbf{l}_p$ and $\mathbf{L}_p$, we can assume $B = -A$ and $B^* = -A^*$. Finally, it is well known (see e.g. [7], p. 411) that since A^* is never zero, there is a linear isometry of $\mathbf{L}_p$, taking A^* to a constant function. Composing f with this isometry we can assume A^* is itself a constant function.

5. Geometry of the subsets $M \subset \mathbf{l}_p$ and $N \subset \mathbf{L}_p$

Let $h_j(t) = \text{sign}\ \sin 2^j \pi t$, $j = 1, 2, \ldots$, be the Rademacher functions on $[0,1]$. Denote by $H \subset \mathbf{L}_p$ the linear subspace spanned by all the h_j, $j = 1, 2, \ldots$. The $\mathbf{L}_2$-norm $||h||_2$ and the $\mathbf{L}_p$-norm $||h||_p$ of a function $h \in H$ are equivalent by Khinchin's inequality. More precisely, we have (see e.g. [6], p.66) that for $\beta = 1/\sqrt{p+2}$

$$||h||_2 \geq \beta \cdot ||h||_p$$

for every $h \in H$.

Denote by $G \subset \mathbf{L}_p$ the linear subspace of infinite codimension consisting of all functions g satisfying the following equations:

$$\int g = 0, \quad \int hg = 0 \quad \text{for any } h \in H.$$

LEMMA 3: *Under the hypothesis of Lemma 2 and the subsequent reductions, N is a subset of the tube*

$$\text{Tub}(G, r) = \{h \in \mathbf{L}_p : d(h, G) < r\}$$

with axis G and radius $r = \delta/L^2 \cdot ||A^*||$.

Proof: For given $C \in N$ we represent the vector C as $(\lambda + g + h)||A^*||$, where λ is a constant, $g \in G$, and $h \in H$. Without loss of generality we can assume that $\lambda \geq 0$. By assumption $C \in N$, this means that

$$||1 + \lambda + g + h|| = ||A^* + C||/||A^*|| < 1 + 2\delta^p .$$

But

$$||1 + \lambda + g + h|| \geq ||1 + \lambda + g + h||_2 \geq ||1 + h||_2 = (1 + ||h||_2^2)^{1/2} .$$

Hence

$$||h||_2 < 3\delta^{p/2}, \quad \text{and} \quad ||h|| < 3\delta^{p/2}/\beta < 3p\delta^{p/2}.$$

Similarly

$$||1 + \lambda + g + h|| \geq ||1 + \lambda + g + h||_2 \geq ||1 + \lambda||_2 = 1 + \lambda.$$

Hence, $\lambda < 2\delta^p$. Thus

$$d(C, G) \leq ||\lambda + h|| \cdot ||A^*|| \leq (\lambda + ||h||)||A^*|| \leq (3p + 2)\delta^{p/2}||A^*|| < \delta/L^2 \cdot ||A^*||$$

by the choice of δ. ∎

Let V be the subspace of $\mathbf{l}_p$ of all sequences supported in the complement of the finite support of the vector A.

LEMMA 4: *The set $M \subset l_p$ contains the intersection of the subspace V and the ball $\mathcal{B}(0, \delta \cdot ||A||)$.*

Proof: The proof is evident. ∎

6. Topological considerations

(The essential ideas for this section are due to J. Donin.)

LEMMA 5: *Let G be a subspace of a Banach space X of infinite codimension, and let γ be a map from the finite-dimensional simplex Δ into X. Then for every $\tau > 0$, there is a continuous map $\gamma_1\colon \Delta \to X$ with $||\gamma_1(x) - \gamma(x)|| = 2\tau$ for every $x \in \Delta$, and so that $\gamma_1(\Delta) \cap \mathrm{Tub}(G, \tau) = \emptyset$.*

Proof: Let F be a finite subset of Δ, so that $\gamma(F)$ is a $\tau/4$-net in $\gamma(\Delta)$. As G is of infinite codimension, so is $\mathrm{sp}\{\gamma(F), G\}$, the subspace spanned by G and $\gamma(F)$. We can thus find a norm one vector $z \in X$, whose distance from $\mathrm{sp}\{\gamma(F), G\}$ is more than 3/4. Define $\gamma_1(x) = \gamma(x) + 2\tau z$.

If $\gamma_1(\Delta) \cap \mathrm{Tub}(G, \tau) \neq \emptyset$, choose $x \in \Delta$ so that $\gamma(x) + 2\tau z = g + y$ for some $g \in G$ and $||y|| < \tau$, and choose $f \in F$ so that $||\gamma(x) - \gamma(f)|| < \tau/4$.

But then

$$\begin{aligned}\mathrm{dist}(z, \mathrm{sp}\{\gamma(F), G\}) &\leq ||z - (g - \gamma(f))/2\tau|| \leq ||y||/2\tau + ||\gamma(x) - \gamma(f)||/2\tau \\ &\leq (\tau + \tau/4)/2\tau = 5/8 < 3/4\end{aligned}$$

contradicting the choice of z. ∎

LEMMA 6: *Let V be a subspace of finite codimension in the Banach space X. Let P be a projection from X onto V, and put $F = \mathrm{Ker}(P)$.*

Fix $\tau > 0$, and put $\Delta = \mathcal{B}_F(\tau)$, the ball of radius τ in F. Then for any continuous map $\gamma\colon \Delta \to X$ satisfying $||\gamma(b) - b|| \leq \tau/||I - P||$ for all $b \in \Delta$, there is a point $v \in \gamma(\Delta) \cap V$. (And clearly $||v|| \leq 2\tau$.)

Proof: By our assumption, the map $\Phi = (I - P)\circ\gamma\colon \Delta \to F$ satisfies $||\Phi(b) - b|| \leq \tau$ for all $b \in \Delta$. By Brouwer's Theorem it follows that there is a $b \in \Delta$ so that $\Phi(b) = 0$. (Take a fixed point of $\vartheta(x) = x - \Phi(x)\colon \Delta \to \Delta$.)

It follows that $v = \gamma(b) \in V$ as required. ∎

Remarks: (1) The notation $\Delta = \mathcal{B}_F(\tau)$ is consistent. It is clearly homeomorphic to a finite-dimensional simplex.

(2) We shall use the lemma when $X = \mathbf{l}_p$, and V is the space of all sequences supported in the complement of some fixed finite set. In this case the cannonical projection satisfies $||P|| = ||I - P|| = 1$. Thus the condition of the lemma is just $||\gamma(b) - b|| \leq \tau$.

7. Proof of Lemma 2

Let V be as in Lemma 4, and let F be the finite-dimensional space of all sequences supported in the support of A. By Lemma 6 (see Remark 2), if $\gamma\colon \Delta = \mathcal{B}_F(\delta||A||) \to \mathbf{l}_p$ is any map with $||x - \gamma(x)|| \leq \delta||A||$ for all $x \in \Delta$, then $\mathrm{Im}\gamma$ intersects $V \cap \mathcal{B}(\delta||A||)$. Hence by Lemma 4, $\mathrm{Im}\gamma \bigcap M \neq \emptyset$ for each such γ.

Assume that $f(M) \subset N$. $f|_\Delta$ is a mapping of Δ into $\mathbf{L}_p$. From Lemma 3 and Lemma 5 it follows that there exists a mapping $\gamma_1\colon \Delta \to \mathbf{L}_p$ such that $\mathrm{Im}\gamma_1$ does not intersect N, and for any $x \in \Delta$,

$$||\gamma_1(x) - f(x)|| < \delta \cdot ||A^*||/L^2 < \delta \cdot ||A||/L. \tag{7.1}$$

For $\gamma = f^{-1} \circ \gamma_1$ we prove now that $||\gamma(x) - x|| < \delta||A||$ for all $x \in \Delta$. This holds, because otherwise there would be such $x \in \Delta$ that

$$||\gamma(x) - x|| \geq \delta||A|| \geq \delta \cdot R/2 \geq 1$$

by the choice of R, and therefore

$$||\gamma_1(x) - f(x)|| = ||f \circ \gamma(x) - f(x)|| > ||\gamma(x) - x||/L \geq \delta \cdot ||A||/L,$$

which contradicts (7.1). Thus we proved that $||\gamma(x) - x|| < \delta||A||$. As noticed above, it follows that $\mathrm{Im}\gamma_1$ must intersect M. This contradicts the assumption that $f(M) \subset N$ and the contradiction proves the lemma. ∎

8. Remark on the case $p < 2$

This technique works for $p < 2$ (which is, in fact, easier) too. In this case the scheme of the proof is the following. We assume that there exists a DLL-homeomorphism $f\colon \mathbf{l}_p \to \mathbf{L}_p$, and get a contradiction. We are interested in the subset $M \subset \mathbf{L}_p$, consisting of all $c\delta^2$-midpoints of A, B, and $N \subset \mathbf{l}_p$, consisting of all $2c\delta^2$-midpoints of A^*, B^* (for a suitable $c = c(p)$). N is a subset of a small neighbourhood of the finite-dimensional linear subspace, consisting of all sequences supported in the finite support of A^* and B^*. M contains the intersection of the ball $\mathcal{B}(\delta\|A\|)$ and the infinite-dimensional subspace H. It follows easily that $f(M) \not\subset N$.

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